ON THE EXISTENCE OF PHASE TRANSITION FOR ONE DIMENSIONAL *P*-ADIC COUNTABLE STATE POTTS MODEL

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In the present paper we shall consider countable state p-adic Potts model on Z_+. A main aim is to establish the existence of the phase transition for the model. In our study, we essentially use one dimensionality of the model. To establish the phase transition we investigation of an infinite-dimensional nonlinear equation. We find a condition on weights to show that the derived equation has two solutions, which yields the existence of the phase transition. Note that it turns out that the finding condition does not depend on values of the prime p, and therefore, an analogous fact is not true when the number of spins is finite.

Keywords: p-adic numbers; countable state; Potts model; p-adic Gibbs measure; weight; phase transition.

1. INTRODUCTION

Due to the assumption that p-adic numbers provide a more exact and more adequate description of microworld phenomena, starting the 1980s, various models described in the language of p-adic analysis have been actively studied (see for exmple [1],[3]). The well-known studies in this area are primarily devoted to investigating quantum mechanics models using equations of mathematical physics [11]. One of the first applications of p-adic numbers in quantum physics appeared in the framework of quantum logic in [2]. This model is especially interesting for us because it could not be described by using conventional real valued probability. Besides, it is also known that a number of p-adic models in physics cannot be described using ordinary Kolmogorov's probability theory. New probability models, namely p-adic values ones were investigated in [5]. Using that p-adic measure theory in [4],[8] the theory of stochastic processes with values in p-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values has been developed [6]. In particular, a non-Archimedean analog of the Kolmogorov theorem was proved. Such a result allows us to construct wide classes of stochastic processes. Therefore, this result gives us a possibility to develop the theory of statistical mechanics in the context of the p-adic theory, since it lies on the basis of the theory of probability and stochastic processes. The this paper we study one-dimensional countable state of nearest-neighbor Potts models over p-adic filed. We are especially interested in the existence of phase transition for the mentioned model. It is worth to mention that when the number of states of the model is finite, say q, then the corresponding p-adic q-state Potts models have been studied in [9,10]. It was established that a strong phase transition occurs if q is divisible by p. This shows that the transition depends on the number of spins q. Therefore, it is interesting to know the situation in the setting with countable states. In [7] first steps to investigation of such a countable state p-adic Potts model on Cayley tree have been studied. We provided a sufficient condition for the uniqueness of p-adic Gibbs measures.

2. PRELIMINARIES

Throughout the paper p will be a fixed prime number greater than 3, i.e. $p \geq 3$. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in Z$, m is a positive integer, $(p,n)=1$, $(p,m)=1$. The p-adic norm of x is given by $|x|_p = p^{-r}$, if $x \neq 0$, and $|x|_p = 0$ for $x = 0$. Such a norm satisfies the following strong triangle inequality $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, this is a non-Archimedean norm. The completion of the field of rational numbers Q with respect to the p-adic norm is called p*-adic field* and it is denoted by Q_p. Given $a \in Q_p$ and $r > 0$ put $B(a,r) = \{x \in Q_p : |x-a|_p < r\}$. The p*-adic exponential* is defined by $\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, which converges for $x \in B(0, p^{-1/(p-1)})$.

Let (X, B) be a measurable space, where B is an algebra of subsets X. A function $\mu : B \to Q_p$ is said to be a p-adic measure if for any $A_1, \ldots, A_n \subset B$ such that $A_i \cap A_j = \varnothing$ $(i \neq j)$ the equality holds $\mu(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j)$. A p-adic measure is called a probability measure if $\mu(X) = 1$. A p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in B\} < \infty$.

In the sequel we will use the notation $Z_+ = \{0,1,2,\cdots\}$. Now define the p-adic Potts model on Z_+ with spin values in the set $\Phi = \{0,1,2,\cdots,\}$. Note that a configuration σ on Z_+ is defined as a function $x \in Z_+ \to \sigma(x) \in \Phi$; in a similar manner one defines configurations σ_n and $\omega_{(n)}$ on $[0,n]$ and $\{n\}$, respectively. The set of all configurations on Z_+ (resp. $[0,n]$, $\{n\}$) coincides with $\Omega = \Phi^{Z_+}$ (resp. $\Omega_n = \Phi^{[0,n]}, \Omega_{\{n\}} = \Phi$). One can see that $\Omega_n = \Omega_{n-1} \times \Phi$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{n-1}$ and $\omega \in \Omega_{\{n\}}$ we define their concatenations by

$$\sigma_{n-1} \vee \omega = \{\{\sigma_{n-1}(k), k \in [0, n-1]\}, \{\omega\}\}.$$

The Hamiltonian $H_n : \Omega_n \to Q_p$ of p-adic countable state Potts model has the form

$$H_n(\sigma) = J\sum_{k=0}^{n-1} \delta_{\sigma(k),\sigma(k+1)}, n \in N, \tag{1}$$

here $\sigma \in \Omega_n$, δ is the Kronecker symbol and $|J|_p \leq 1/p$.

Let us construct p-adic Gibbs measures corresponding to the model. A given set A we put $Q_p^A = \{\{x_i\}_{i\in A} : x_i \in Q_p\}$. Assume that a function $\mathbf{h} : N \to Q_p^{\Phi}$, i.e. $\mathbf{h}_n = \{h_{i,n}\}_{i\in\Phi}$, $n \in N$ is given and a non-zero element $\lambda = \{\lambda(i)\}_{i\in\Phi} \in Q_p^{\Phi}$ is fixed such that $|\lambda(n)|_p \to 0$ as $n \to \infty$, which is called a *weight*. In what follows, without losing generality we may assume that $\lambda(0) \neq 0$.

Given $n = 1, 2, \ldots$ a p-adic probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_n is defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp_p\{H_n(\sigma)\} h_{\sigma(n),n} \prod_{k=0}^{n} \lambda(\sigma(k)), \tag{2}$$

here, $\sigma \in \Omega_n$ and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor called a *partition function* given by

$$Z_n^{(\mathbf{h})} = \sum_{\sigma\in\Omega_n} \exp_p\{H_n(\sigma)\} h_{\sigma(n),n} \prod_{k=0}^{n} \lambda(\sigma(k)).$$

Note that the measures $\mu_{\mathbf{h}}^{(n)}$ are well defined.

In the paper we want to define a p-adic probability measure μ on Ω such that it would be compatible with defined ones $\mu_{\mathbf{h}}^{(n)}$, i.e.

$$\mu(\sigma \in \Omega : \sigma|_{[0,n]} = \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \text{ for all } \sigma_n \in \Omega_n, n \in N.$$

In general, à priori the existence of such a kind of measure μ is not known, since, there is not much information on topological properties. Therefore, at a moment, we can only use the p-adic Kolmogorov extension Theorem ([6]) which based on so called *compatibility condition* for the measures $\mu_{\mathbf{h}}^{(n)}$, $n \geq 1$, i.e. $\sum_{\omega\in\Phi} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1})$, for any $\sigma_{n-1} \in \Omega_{n-1}$. This condition according to the theorem implies the existence of a unique p-adic measure μ defined on Ω with a required condition. So, if for some function $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ satisfy the compatibility condition, then there is a unique p-adic probability measure, which we denote by $\mu_{\mathbf{h}}$, since it depends on $\mathbf{h}$. Such a measure $\mu_{\mathbf{h}}$ is said to be *a generalized p-adic Gibbs measure* corresponding to the p-adic Potts model. By

$GG(H)$ we denote the set of all generalized p-adic Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_n, n \in N\}$. If $|GG(H)| \geq 2$ (here $|A|$ stands for the cardinality of a set A) then we say that *a phase transition* occurs for the model, otherwise, there is *no phase transition*. If the function $\mathbf{h}$ has a special form, i.e. $\mathbf{h} = \{\exp_p(\kappa_{i,n})\}_{i\in\Phi}$ for some $\{\kappa_{i,n}\} \subset Q_p$, then the corresponding measure defined by (2) is called *p-adic Gibbs measure*. The set of all p-adic Gibbs measures is denoted by $G(H)$. If $|G(H)| \geq 2$, then we say that for this model there exists *a strong phase transition*. Now one can ask for what kind of functions $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ defined by (2) would satisfy the compatibility condition. The following theorem gives an answer to this question.

Theorem 2.1. [7] *The measures $\mu_{\mathbf{h}}^{(n)}$, $n = 1, 2, \ldots$ (see (2)) satisfy the compatibility condition (10) if and only if for any $n \in N$ the following equation holds:*

$$\hat{h}_{i,n} = \frac{\lambda(i)}{\lambda(0)} F_i(\hat{\mathbf{h}}_{n+1}; \theta), i \in N \tag{3}$$

here and below $\theta = \exp_p(J)$, a vector $\hat{\mathbf{h}} = \{\hat{h}_i\}_{i\in N} \in Q_p^N$ is defined by a vector $\mathbf{h} = \{h_i\}_{i\in\Phi}$ as follows

$$\hat{h}_i = \frac{h_i \lambda(i)}{h_0 \lambda(0)}, i \in N \tag{4}$$

and mappings $F_i : Q_p^N \times Q_p \to Q_p$ are defined by

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{\infty} x_j + 1}{\sum_{j=1}^{\infty} x_j + \theta}, \mathbf{x} = \{x_i\}_{i\in N}, i \in N. \tag{5}$$

Let us recall that a function $\{\mathbf{h}_n\}_{n\in N}$ is translation-invariant if $\mathbf{h}_n = \mathbf{h}_{n+1} := \mathbf{h}$ for every $n \in N$. It is natural to ask is there a translation invariant solution of (3).

Now we are looking for the translation-invariant solution $\hat{\mathbf{h}}$ of (3). Then the equation can be written as follows

$$\hat{h}_i = \frac{\lambda(i)}{\lambda(0)} \left(\frac{(\theta - 1)\hat{h}_i + \sum_{j=1}^{\infty} \hat{h}_j + 1}{\sum_{j=1}^{\infty} \hat{h}_j + \theta} \right), i \in N. \tag{6}$$

Investigating, the derived equation (6), we have proved the following

Theorem 2.2. [7] *Let $0 < |J|_p < p^{-1/(p-1)}$ and for the weight λ the condition*

$$\lambda(0) = 1, \text{and } |\lambda(m)|_p < 1 \qquad \forall m \in N. \tag{7}$$

be satisfied. Then for one dimensional p-adic Potts model (1) there is a generalized p-adic Gibbs measure, i.e. $|GG(H)| \geq 1$. Moreover, there is a unique p-adic Gibbs measure, i.e. $|G(H)| = 1$.

3. MAIN RESULTS

In this section we are going to show that the equation (3) has at least two translation-invariant solutions under some conditions. In this section we will assume the following

$$\lambda(0)=1, \lambda(1)=\alpha, \text{and } |\lambda(m)|_p<1 \qquad \forall m\geq 2, \tag{8}$$

here $\alpha\in Q_p$ such that

$$|\alpha|_p=1, |1-\alpha|_p\leq 1/p. \tag{9}$$

It is obvious that in this case (7) is not satisfied. Now we are going to find translation invariant solution of (3). Therefore, we assume that $\hat{\mathbf{h}}_1=(x_1,\dots,x_n,\dots)$. Let us for the sake of shortness, a given sequence $\mathbf{x}=\{x_j\}_{j\geq 2}$ we denote $X:=\sum_{j=2}^{\infty}x_j$. If $\hat{\mathbf{h}}_1$ is a translation invariant solution, then the first equation in (6) with (8) can be reduced to $P(x_1)=0$, where $P(x)=x^2+(X+\theta(1-\alpha))x-\alpha(X+1)$.

If $|X+2|_p=1$, then (9) with the Hensel's Lemma implies that the equation has a solution $x_{\pm,1}$ belonging to Q_p. In the sequel we will need an exact form of these solutions, which can be written as follows

$$x_{\pm,1}=\frac{(\alpha-1)\theta-X\pm\sqrt{D_X}}{2},$$

where $D_X=(X+\theta(1-\alpha))^2+4\alpha(X+1)$.

Note that the existence of the solutions $x_{\pm,1}$ yields the existence $\sqrt{D_X}$. By substituting the above solutions into F_i in (5), we have

$$F_i^{(\pm)}(\mathbf{x};\theta)=\frac{2(\theta-1)x_i+(\alpha-1)\theta+X\pm\sqrt{D_X}+2}{(\alpha+1)\theta+X\pm\sqrt{D_X}}, \quad i\geq 2,$$

where $\mathbf{x}=\{x_i\}_{i\geq 2}$. Note that from (4) we see that $|x_n|_p\to 0$ as $n\to\infty$. Therefore, it is natural to consider the following space $c_0=\{\{x_n\}_{n\geq 2}\subset Q_p : |x_n|_p\to 0, n\to\infty\}$ with a norm $\|x\|=\max_n|x_n|_p$. Define $\mathbf{B}_r=\{\{x_n\}\in c_0 : \|x\|\leq r\}$, where $r\in\{p^k : k\in Z\}$. It is clear that $\mathbf{B}_{r_p}$ is a closed subset of c_0. Now consider the following mapping

$$(F^{(\pm)}(\mathbf{x}))_i=\lambda(i)F_i^{(\pm)}(\mathbf{x},\theta), \quad i\geq 2,$$

where $\mathbf{x}=\{x_n\}\in c_0$. Now our aim is to show the existence of a fixed point of $F^{(\pm)}$. Put $\delta=\max_{i\geq 2}|\lambda(i)|_p$. From (8) one immediately finds that $\delta<1$. Note that according to the condition (8) from (4) we obtain $|x_n|_p \le |\lambda(n)|_p$, $\forall n \ge 2$, which implies that any solution of (3) belongs to $\mathbf{B}_\delta$. This means $F^{(\pm)}(\mathbf{B}_\delta) \subset \mathbf{B}_\delta$. One can prove that $\| F^{(\pm)}(\mathbf{x}) - F^{(\pm)}(\mathbf{y}) \| \le \delta |\theta - 1|_p \| \mathbf{x} - \mathbf{y} \|$, for every $\mathbf{x}, \mathbf{y} \in \mathbf{B}_\delta$. Using this we are able to prove our main result.

Theorem 3.1. *Let the conditions (8),(9) be satisfied. Then a phase transition occurs for the countable state p-adic Potts model (1).*

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